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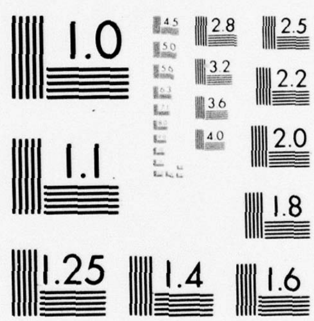


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# PHASE SPACE PATH INTEGRALS, WITHOUT LIMITING PROCEDURE

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## ABSTRACT

This paper defines path integrals in phase space without using a time-division approach followed by a limiting process, thereby generalizing a similar procedure used in configuration space. This is useful since the path integral approach cannot always be formulated in configuration space (e.g., when the Hamiltonian is arbitrary) but can always be formulated in phase space. The most general Gaussian measure, absorbing the quadratic portion of the functional to be integrated, is constructed, and large classes of path integrals are evaluated with respect to it. Applications are given to the perturbation expansion and the semiclassical (WKB) expansion for arbitrary Hamiltonians.

# I. INTRODUCTION

The quantum-mechanical propagator $\langle q_b, t_b | q_a, t_a \rangle$, or probability amplitude that a particle at position $q_a$ at time $t_a$ will be at position $q_b$ at time $t_b$, can be written as a phase space path integral:

$$\langle q_b, t_b | q_a, t_a \rangle \equiv K = \int_{\mathcal{P}} \left[\frac{dp\,dq}{2\pi\hbar}\right] \exp \frac{i}{\hbar} \int_{t_a}^{t_b} \{ p(t)\dot{q}(t) - H[p(t), q(t), t] \} dt, \quad (1)$$

where $H$ is the classical Hamiltonian of the system, or a quantity suitably related to it[1], and $\mathcal{P}$ is the space of phase space paths $(q, p)$ satisfying $q(t_a) = q_a$ and $q(t_b) = q_b$, with $p(t)$ unrestricted. The integral is usually defined by the time division procedure[2], i.e.

$$K = \lim_{\substack{m\to\infty \\ \max_j (t_{j+1}-t_j)\to 0}} \int_{\mathbb{R}^{2m+1}} \frac{dq_1 \ldots dq_m \, dp_0 \, dp_1 \ldots dp_m}{(2\pi\hbar)^{m+1}} \cdot \quad (2)$$

$$\times \exp \frac{i}{\hbar} \left\{ \sum_{j=0}^{m} \left[ p_j \cdot \left( \frac{q_{j+1} - q_j}{t_{j+1} - t_j} \right) - H\left( p_j, \frac{q_{j+1} + q_j}{2}, t_j \right) (t_{j+1} - t_j) \right\}$$

with $q_{m+1} \equiv q_b$, $q_0 \equiv q_a$, $t_0 \equiv t_a$, and $t_{m+1} \equiv t_b$. We work in one dimension to simplify the discussion. The results can be easily generalized. The Einstein summation convention over repeated indices is used throughout.

The limiting process makes the scheme difficult to use for computational purposes, not to mention questions of mathematical legality. It has been done away with in the case of the Wiener functional integral[3], and the method was later extended to Feynman path integrals in the configuration space of quantum mechanics[4-9]. The new formalism rests on defining what plays the role of a measure in path space by its Fourier transform, which is a simple closed-form expression. This is all that is needed to completely define the object and reduce many path integrals to ordinary definite integrals[10]. We do not treat the mathematical problems here, as we are mainly concerned with developing computational techniques.

The purpose of this paper is to extend this limiting-procedure-free formalism to phase space. This is necessary not only from the point of view of completeness, but also because phase space path integrals are more basic than configuration space path integrals. Indeed, the latter provide a solution to the Schrödinger equation only for Hamiltonian operators quadratic in the momenta, whereas the former apply to arbitrary Hamiltonian operators[6,11], a useful generalization.

After constructing the most general Gaussian measure in phase space, we evaluate large classes of path integrals with respect to it, and present applications to the perturbation expansion and the semiclassical expansion for arbitrary Hamiltonians.

## II. CONSTRUCTION OF THE PHASE SPACE MEASURE

We wish to construct the most general Gaussian measure $w(p,q)$ in phase space, the one which will absorb the entire quadratic term in the functional to be integrated. To be more specific, this measure will be equivalent to:

$$dw(p,q) \sim \frac{1}{K_0}\left[\frac{dp\,dq}{2\pi\hbar}\right]\exp\frac{i}{\hbar}\int_{t_a}^{t_b}\left\{p(t)\dot{q}(t) - H_0\left[p(t),q(t),t\right]\right\}dt \qquad (3)$$

where

$$H_0(p,q,t) = g(t)\frac{p^2}{2m} + \frac{1}{2}f(t)q^2 + k(t)pq \qquad (4)$$

and $K_0$ is the normalization factor, ensuring that:

$$\int_{\mathcal{P}} dw(p,q) = 1 \qquad (5)$$

It is readily observed that $K_0$ must be the propagator associated with the Hamiltonian $H_0$. The functions $g(t)$, $f(t)$, and $k(t)$ depend on the problem investigated. If one wishes to write a path integral for a Hamiltonian of the form $H_0 + \alpha H_1$, where $H_1$ contains the terms beyond quadratic, then the measure $w$ enables one to obtain the propagator as a perturbation expansion in powers of $\alpha$. If, in a more useful application, one first expands the action functional about the classical position and classical momentum, $q_c(t)$ and $p_c(t)$, then the measure $w$ yields the terms of a semiclassical (WKB) expansion of the propagator. The functions $g$, $f$, and $k$ then contain $q_c(t)$ and $p_c(t)$. All this will be further examined below.

A proper way to define (and use) $w$ without the time-slicing procedure that (3) entails is to build its Fourier transform. The Fourier transform of $w$ can be written as:

$$\mathcal{F}w(\mu,\nu) = \int_{\mathcal{P}} e^{-i\langle\mu,q\rangle - i\langle\nu,p\rangle}\, dw(p,q), \tag{6}$$

where $\mu$ and $\nu$ are elements of $\mathcal{M}$, the space of bounded measures on the time interval $T \equiv [t_a, t_b]$. For example, if $\mu$ is induced by a function, i.e. $d\mu(t) = f(t)dt$, then

$$\langle\mu,q\rangle \equiv \int_T q(t)\,d\mu(t)\ ; \tag{7}$$

if $\mu$ is $\delta_s$, the delta function at $s$, then

$$\langle\delta_s,q\rangle \equiv q(s). \tag{8}$$

The fundamental observation is that if we put $d\mu(t) = B(t)/\hbar$ and $d\nu(t) = A(t)/\hbar$, then the Fourier transform (6) is nothing other than $K/K_0$, where $K$ is the propagator corresponding to the auxiliary Hamiltonian

$$H(p,q,t) = g(t)\frac{p^2}{2m} + \frac{1}{2}f(t)q^2 + k(t)pq + A(t)p + B(t)q. \tag{9}$$

Both $K$ and $K_0$ can be calculated exactly given the associated classical paths. Indeed, since both correspond to quadratic Hamiltonians, their semiclassical (WKB) approximations are exact. The latter are given by:

$$K_{WKB} = \left(\frac{M}{2\pi i \hbar}\right)^{1/2} \exp\left(\frac{i S_c}{\hbar}\right), \tag{10}$$

where $S_c$ is the action functional evaluated at the classical position and momentum $q_c$ and $p_c$ , and $M$ is the Van Vleck - Morette function $-\partial^2 S_c / \partial q_a \partial q_b$. Thus, the problem of determining the phase space measure $w$ reduces to solving the classical problem for $H$ and $H_0$. Note that the quantum operators corresponding to the $pq$ terms in $H$ and $H_0$ are the symmetrized[12] $\frac{1}{2}(\underset{\sim}{P}\underset{\sim}{Q} + \underset{\sim}{Q}\underset{\sim}{P})$.

We first state the main theorem, then we prove it.

THEOREM 1. The normalized Gaussian measure $w(p,q)$ in phase space $\mathcal{P}$ corresponding to

$$dw(p,q) \sim \frac{1}{K_0}\left[\frac{dp\,dq}{2\pi\hbar}\right] \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left[p(t)\dot{q}(t) - \frac{1}{2m}g(t)p^2(t) - \frac{1}{2}f(t)q^2(t) - k(t)p(t)q(t)\right]dt\right\} \tag{11}$$

has the following Fourier transform[13]:

$$\mathcal{F}w(\mu,\nu) = \exp\left\{-i\langle\mu,\bar{q}\rangle - i\langle\nu,\bar{p}\rangle - \frac{i\hbar}{2}\int_T\int_T G_{ab}(t,t')\,d\mu(t)\,d\mu(t') - i\hbar\int_T\int_T \bar{G}(t,t')\,d\mu(t)\,d\nu(t') - \frac{i\hbar}{2}\int_T\int_T G_p(t,t')\,d\nu(t)\,d\nu(t')\right\} \tag{12}$$

$$\equiv \exp\left\{-i\int_T \bar{r}(t)\,d\tilde{\alpha}(t) - \frac{i\hbar}{2}\int_T\int_T d\alpha(t)\,\mathcal{G}(t,t')\,d\tilde{\alpha}(t')\right\} \tag{13}$$

where

(1) $$\mathcal{P} \equiv \left\{[p(t),q(t)] \text{ on } T \equiv [t_a,t_b] \;\middle|\; q(t_a)=q_a,\ q(t_b)=q_b,\ p(t) \text{ unrestricted}\right\} \tag{14}$$

(2) the normalization factor $K_0$ is the propagator corresponding to the Hamiltonian:

$$\underset{\sim}{H}_0 = \frac{g(t)}{2m}\underset{\sim}{P}^2 + \frac{1}{2}f(t)\underset{\sim}{Q}^2 + \frac{1}{2}k(t)(\underset{\sim}{P}\underset{\sim}{Q} + \underset{\sim}{Q}\underset{\sim}{P}), \tag{15}$$

for which the WKB approximation is exact.

(3) $$d\alpha(t) \equiv (d\mu(t) \quad d\nu(t)) \; ; \; d\tilde{\alpha}(t) \equiv \begin{pmatrix} d\mu(t) \\ d\nu(t) \end{pmatrix}. \tag{16}$$

(4) $\bar{r}(t) \equiv (\bar{q}(t), \bar{p}(t))$ , the average path in $\mathcal{P}$ with respect to the measure $w$

$$= (q_{co}(t), p_{co}(t)), \tag{17}$$

where $q_{co}$ and $p_{co}$ are the classical position and momentum corresponding to $H_0$ . They are related by:

$$p_{co}(t) = \frac{m}{g(t)}\left[\frac{d}{dt} - k(t)\right]q_{co}(t). \tag{18}$$

(5)

$$\mathcal{G}(t,t') \equiv \begin{pmatrix} G_{ab}(t,t') & \bar{G}(t,t') \\ \bar{G}(t',t) & G_p(t,t') \end{pmatrix} \tag{19}$$

is a Green function of the small disturbance operator in phase space corresponding to $H_0$ :

$$\underset{\sim}{\mathcal{O}} = \begin{pmatrix} -f(t) & -k(t) - \frac{d}{dt} \\ -k(t) + \frac{d}{dt} & -\frac{1}{m}g(t) \end{pmatrix}, \tag{20}$$

i.e. $$\underset{\sim}{\mathcal{O}}\,\mathcal{G}(t,t') = \delta(t-t')\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{21}$$

$\mathcal{G}(t,t')$ is independent of $q_a$ and $q_b$ .

(6) $G_{ab}(t,t')$ is the (symmetric) Green function of the small disturbance operator in configuration space which vanishes at both endpoints:

$$\underset{\sim}{\mathcal{S}} \equiv \frac{-m}{g(t)}\left[\frac{d^2}{dt^2} - \frac{\dot{g}(t)}{g(t)}\frac{d}{dt} - \dot{k}(t) + \frac{1}{m}f(t)g(t) - k^2(t) + \frac{\dot{g}(t)k(t)}{g(t)}\right], \tag{22}$$

i.e.

$$\underset{\sim}{\mathcal{S}}\, G_{ab}(t,t') = \delta(t-t');\ G_{ab}(t,t') = G_{ab}(t',t);\ G_{ab}(t_a,t) = G_{ab}(t_b,t) = 0. \tag{23}$$

(7)

$$\bar{G}(t,t') = \frac{m}{g(t')}\left[\frac{\partial}{\partial t'} - k(t')\right] G_{ab}(t,t') \tag{24}$$

(8)

$$G_p(t,t') = \frac{m^2}{g(t)g(t')}\left[\frac{\partial}{\partial t} - k(t)\right]\left[\frac{\partial}{\partial t'} - k(t')\right] G_{ab}(t,t') - m g^{-1}(t)\,\delta(t-t') \tag{25}$$

The $\delta$ function term in (25) is always cancelled by a similar term. When $t=t'$, $G_{ab}$ and $G_p$ are continuous, but $\bar{G}$ has a jump of magnitude 1:

$$\left(\lim_{t\to t'} - \lim_{t'\to t}\right)\bar{G}(t,t') = 1. \tag{26}$$

Note that the measure $w(p,q)$ does not split the path integral into an integral over momentum space followed by an integral over configuration space, each with its own measure. Thus one truly has a "phase space" path integral. However, the measure $w$ induces in a natural manner measures $w_{ab}$ and $w_p$ on configuration space alone and momentum space alone by

$$\mathcal{F}w_{ab}(\mu) \equiv \mathcal{F}w(\mu,0) \quad \text{and} \quad \mathcal{F}w_p(\nu) = \mathcal{F}w(0,\nu). \tag{27}$$

The measure $w_{ab}$ in the configuration space of paths such that $q(t_a) = q_a$ and $q(t_b) = q_b$ is studied in Ref. 7.

Proof of Theorem 1

The Lagrangian $L_0$ corresponding to $H_0$ in (4) and the Lagrangian $L$ corresponding to the auxiliary $H$ in (9) are:

$$L(q,\dot q,t) = \frac{m}{2g(t)}\left[\dot q - A(t) - k(t)q\right]^2 - \frac{1}{2}f(t)q^2 - B(t)q\,, \tag{28}$$

$$L_0(q,\dot q,t) = \frac{m}{2g(t)}\left[\dot q - k(t)q\right]^2 - \frac{1}{2}f(t)q^2. \tag{29}$$

The classical paths $q_c$ and $q_{c0}$ satisfy the Euler-Lagrange equations:

$$\underset{\sim}{Q}\, q_c(t) = u(t) \tag{30}$$

$$\underset{\sim}{Q}\, q_{c0}(t) = 0\,, \tag{31}$$

where $\underset{\sim}{Q}$ is a second-order linear differential operator:

$$\underset{\sim}{Q} \equiv \frac{d^2}{dt^2} - \frac{\dot g(t)}{g(t)}\frac{d}{dt} - \dot k(t) + \frac{1}{m}f(t)g(t) - k^2(t) + \frac{\dot g(t)}{g(t)}k(t) \tag{32}$$

and $u(t)$ depends on $A(t)$ and $B(t)$ :

$$u(t) \equiv \dot A(t) - \frac{1}{m}B(t)g(t) + k(t)A(t) - A(t)\frac{\dot g(t)}{g(t)}\,. \tag{33}$$

Both classical paths go through $q_a$ at $t_a$ and $q_b$ at $t_b$ . The substitutions

$$q_c(t) = D_c(t)\left[\frac{g(t)}{g(t_a)}\right]^{1/2} \quad\text{and}\quad q_{c0}(t) = D_{c0}(t)\left[\frac{g(t)}{g(t_a)}\right]^{1/2} \tag{34}$$

eliminate the $d/dt$ term in $\underset{\sim}{Q}$ , and replace (30) and (31) by

$$\underset{\sim}{\mathcal{D}}\, D_c(t) = -u(t)\left[\frac{g(t_a)}{g(t)}\right]^{1/2} \tag{35}$$

$$\underset{\sim}{\mathcal{D}}\, D_{c0}(t) = 0\,, \tag{36}$$

where

$$\underset{\sim}{\mathcal{D}} \equiv -\frac{d^2}{dt^2} - w(t) , \tag{37}$$

with

$$w(t) = \frac{1}{2}\frac{\ddot{g}(t)}{g(t)} - \frac{3}{4}\frac{\dot{g}^2(t)}{g^2(t)} + \frac{\dot{g}(t)k(t)}{g(t)} - k^2(t) + \frac{1}{m}f(t)g(t) - \dot{k}(t) \tag{38}$$

Note that

$$\underset{\sim}{Q}\left[\sqrt{g(t)}\, f(t)\right] = -\sqrt{g(t)}\, \underset{\sim}{\mathcal{D}} f(t) . \tag{39}$$

Let $D_1(t)$ and $D_2(t)$ be two solutions of (36), subject to the boundary conditions:

$$D_1(t_b) = 1 \qquad \dot{D}_1(t_b) = 0$$
$$D_2(t_b) = 0 \qquad \dot{D}_2(t_b) = -1 \tag{40}$$

The Wronskian $W = \dot{D}_1 D_2 - D_1 \dot{D}_2$ is constant for equations of the form $\underset{\sim}{\mathcal{D}} D(t) = 0$. In this case the boundary conditions indicate that $W$ is equal to 1. Since $W$ is different from 0, $D_1$ and $D_2$ are linearly independent, and the general solution of (36) is a linear combination of $D_1$ and $D_2$. If we define the antisymmetric kernel $J(t,t')$ by:

$$J(t,t') \equiv D_1(t')D_2(t) - D_1(t)D_2(t') \tag{41}$$

then the classical path $q_{co}$ can be written as

$$q_{co}(t) = \frac{\sqrt{g(t)}}{J(t_a,t_b)}\left[q_a \frac{J(t,t_b)}{\sqrt{g(t_a)}} + q_b \frac{J(t_a,t)}{\sqrt{g(t_b)}}\right] . \tag{42}$$

The classical path $q_c$ can be easily shown to be

$$q_c(t) = q_{co}(t) - \sqrt{g(t)} \int_T \frac{u(s)}{\sqrt{g(s)}}\, G(s,t)\, ds \tag{43}$$

where $G$ is the (symmetric) Green function of $\underset{\sim}{\mathcal{D}}$ which vanishes at both endpoints:

$$\begin{cases} \underset{\sim}{\mathcal{D}}\, G(t,t') = \delta(t-t') \;;\quad G(t,t') = G(t',t) & (44) \\ G(t_a,t) = G(t_b,t) = 0 & (45) \end{cases}$$

This Green function can be built from the solutions $D_1$ and $D_2$ of $\underset{\sim}{\mathcal{D}} D = 0$. It is[6,8,9]:

$$G(t,t') = \frac{J(t_a,t)J(t',t_b)Y(t'-t) + J(t_a,t')J(t,t_b)Y(t-t')}{J(t_a,t_b)}, \quad (46)$$

$Y(t)$ being the Heaviside step function, equal to 1 for $t > 0$ and 0 otherwise. This can be verified by direct substitution. If $u(s)$ is replaced by its expression (33) in terms of $A$ and $B$, and the $\dot{A}$ term is integrated by parts (the integrated term vanishes), then the difference $\xi(t)$ of the classical paths depends linearly on $A$ and $B$ as follows:

$$\xi(t) \equiv q_c(t) - q_{c_0}(t) = \int_T A(s)\,\omega(s,t)\,ds + \int_T B(s)\,\sigma(s,t)\,ds \quad (47)$$

where

$$\omega(s,t) \equiv \sqrt{\frac{g(t)}{g(s)}}\left[\frac{1}{2}\frac{\dot{g}(s)}{g(s)} - k(s) + \frac{\partial}{\partial s}\right] G(s,t) \quad (48)$$

$$\sigma(s,t) \equiv \frac{1}{m}\sqrt{g(t)g(s)}\; G(s,t). \quad (49)$$

As we established earlier, the Fourier transform of the measure $w$ is the ratio $K/K_0$ of the propagators corresponding to $H$ and $H_0$, which in turn happened to be exactly equal to their WKB approximants. If $d\mu(t) \equiv B(t)/\hbar$ and $d\nu(t) \equiv A(t)/\hbar$, then

$$\mathcal{F}w(B,A) = \frac{K}{K_0} = \sqrt{\frac{M}{M_0}}\,\exp\left\{\frac{i}{\hbar}\int_T L(q_c,\dot{q}_c,t)\,dt - \frac{i}{\hbar}\int_T L_0(q_{c0},\dot{q}_{c0},t)\,dt\right\} \quad (50)$$

The Van Vleck - Morette functions $M$ and $M_0$ are equal since $H$ and $H_0$ differ only by terms linear in $p$ and $q$. We give their value for completeness. It is:

$$M = M_0 = \frac{m}{J(t_a,t_b)\sqrt{g(t_a)\,g(t_b)}} \tag{51}$$

This can be easily proved. Indeed,

$$M \equiv -\frac{\partial^2 S_c}{\partial q_a \partial q_b} = \frac{\partial p_c(t_a)}{\partial q_b} \tag{52}$$

since $p_c(t_a) = -\partial S_c/\partial q_a$. The momentum corresponding to the Lagrangian $L$ in (28) is

$$p = \frac{\partial L}{\partial \dot q} = \frac{m}{g(t)}\left[\dot q - A(t) - k(t)q\right], \tag{53}$$

and hence $\partial p_c(t_a)/\partial q_b = m g^{-1}(t_a)\,\partial \dot q_c(t_a)/\partial q_b$. The result can then be easily establised by using (42), (41), and (40), along with the fact that the Wronskian of $D_1$ and $D_2$ is 1.

Substituting (28) and (29) in (50) yields:

$$\begin{aligned} \mathcal{F}_W(B,A) = \exp \frac{i}{\hbar}\Big\{ & \frac{m}{2}\int_T \frac{dt}{g(t)}\left[\dot\xi(t) - k(t)\xi(t) - A(t)\right]\cdot \\ & \times\left[\dot\xi(t) + 2\dot q_{co}(t) - A(t) - k(t)\xi(t) - 2k(t)q_{co}(t)\right] \\ & - \int_T B(t)\left[q_{co}(t) + \xi(t)\right] - \frac{1}{2}\int_T f(t)\xi(t)\left[\xi(t) + 2q_{co}(t)\right]\Big\} \end{aligned} \tag{54}$$

Now substituting for $\xi(t)$ its expression in (47) yields the full explicit dependence of $\mathcal{F}_W(B,A)$ on $A$ and $B$, which is of the form:

$$\mathcal{F}_W(B,A) = \exp \frac{i}{\hbar}\Big\{ -\int_T \bar{q}(t)B(t)dt - \int_T \bar{p}(t)A(t)dt$$
$$-\frac{1}{2}\int_T\int_T G_{ab}(t,t')B(t)B(t')dt\,dt' - \frac{1}{2}\int_T\int_T G_p(t,t')A(t)A(t')dt\,dt'$$
$$-\int_T\int_T \bar{G}(t,t')B(t)A(t')dt\,dt'\Big\} . \tag{55}$$

The various functions entering this expression are calculated below one by one and found to be as given in the statement of the theorem. Since they will involve small disturbance equations, we think it useful to first exhibit these equations.

<u>Equations of small disturbances</u>

The small disturbance equation (or Jacobi equation, or equation of geodesic deviation in the language of curved spaces) is that satisfied by a small deviation from the classical path. Thus, since the Euler-Lagrange (or Hamilton) equations yielding the classical path are obtained by setting the first variation of the action functional equal to zero, the small disturbance equation is obtained by setting the second variation of the action equal to zero. For Lagrangian actions, it is

$$\underset{\sim}{\mathcal{S}}\,\alpha(t) = 0 \tag{56}$$

where $\underset{\sim}{\mathcal{S}}$ is the small disturbance operator in configuration space:

$$\underset{\sim}{\mathcal{S}} = \left\{ -\frac{\partial^2 L}{\partial \dot{q}^2}\frac{d^2}{dt^2} - \left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}^2}\right)\right]\frac{d}{dt} + \frac{\partial^2 L}{\partial q^2} - \left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial q\,\partial \dot{q}}\right)\right]\right\}_{q=q_c} \tag{57}$$

For Hamiltonian actions, it is

$$\underset{\sim}{\mathcal{O}}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{58}$$

where $\underset{\sim}{\mathcal{O}}$ is the small disturbance operator in phase space:

$$\underset{\sim}{\mathcal{O}} = \begin{pmatrix} -\dfrac{\partial^2 H}{\partial q^2} & -\dfrac{\partial^2 H}{\partial q \partial p} - \dfrac{d}{dt} \\ -\dfrac{\partial^2 H}{\partial p \partial q} + \dfrac{d}{dt} & -\dfrac{\partial^2 H}{\partial p^2} \end{pmatrix}_{\substack{q=q_c \\ p=p_c}} . \tag{59}$$

In the case of $H_0$, $\underset{\sim}{\mathcal{S}}$ and $\underset{\sim}{\mathcal{O}}$ are given by (22) and (20). An interesting observation: the elements of $\underset{\sim}{\mathcal{O}}$ can be used to form $\underset{\sim}{\mathcal{S}}$ as follows:

$$\underset{\sim}{\mathcal{S}} = -f(t) + \left[k(t) + \frac{d}{dt}\right]\left[\frac{m}{g(t)}\right]\left[k(t) - \frac{d}{dt}\right] . \tag{60}$$

Note also that $\underset{\sim}{\mathcal{S}}$ and $\underset{\sim}{Q}$ are related by

$$\underset{\sim}{\mathcal{S}} = \frac{-m}{g(t)} \underset{\sim}{Q} \tag{61}$$

Calculation of the elements of the measure $\mathcal{W}$

All the calculations below involve integrations by parts where the integrated term vanishes due to (45). The comma denotes differentiation with respect to the variable indicated. Thus with reference to (55), we have:

The BB term

$$\begin{aligned} G_{ab}(t,t') &= 2\sigma(t,t') + \int_T \left[f(s) - \frac{m k^2(s)}{g(s)}\right] \sigma(t,s)\sigma(t',s)\,ds \\ &\quad - m\int_T \frac{ds}{g(s)} \sigma_{,s}(t,s)\sigma_{,s}(t',s) + m\int_T \frac{ds}{g(s)} k(s)\left[\sigma(t,s)\sigma(t',s)\right]_{,s} \\ &= 2\sigma(t,t') + \int_T ds\, \sigma(t,s) \left\{ f(s) + m\frac{\partial}{\partial s}\left[\frac{1}{g(s)}\frac{\partial}{\partial s}\right] \right. \\ &\quad \left. - m\left[\frac{d}{ds}\frac{k(s)}{g(s)}\right] - m\frac{k^2(s)}{g(s)} \right\} \sigma(t',s) \end{aligned} \tag{62}$$

The operator between curly brackets can be easily shown to be $m g^{-1}(t) \underset{\sim}{Q}$ i.e. $-\underset{\sim}{\mathcal{S}}$. From (39), (45), and (49) we can establish the relation:

$$\underset{\sim}{\mathcal{S}}_t \, \sigma(u,t) = \delta(u-t). \tag{63}$$

Substituting in (62), we have

$$G_{ab}(t,t') = \sigma(t,t') \equiv m^{-1}\sqrt{g(t)g(t')}\; G(t,t'). \tag{64}$$

Therefore, $G_{ab}(t,t')$ is indeed a Green function of $\underset{\sim}{\mathcal{S}}$ which vanishes at $t_a$ and $t_b$. Since it is symmetric, it is continuous along the diagonal $t = t'$. Q.E.D.

The AB term

$$\bar{G}(t,t') = \omega(t',t) + \frac{m}{g(t')}\left[\frac{\partial}{\partial t'} - k(t')\right]\sigma(t,t')$$
$$+ \int_T \sigma(t,s)\omega(t',s)\left[f(s) - \frac{m k^2(s)}{g(s)}\right]ds + m\int_T \frac{ds}{g(s)}\, k(s) \cdot$$
$$\times \left[\omega(t',s)\sigma(t,s)\right]_{,s} - m\int_T \frac{ds}{g(s)}\, \omega_{,s}(t',s)\sigma_{,s}(t,s) \tag{65}$$
$$= \omega(t',t) + \frac{m}{g(t')}\left[\frac{\partial}{\partial t'} - k(t')\right]\sigma(t,t') - \int_T \omega(t',s)\, \underset{\sim}{\mathcal{S}}_s\, \sigma(t,s) \tag{66}$$
$$= \frac{m}{g(t')}\left[\frac{\partial}{\partial t'} - k(t')\right] G_{ab}(t,t') \tag{67}$$

in view of (64) and (23). Q.E.D.

Using the specific expression [(64) with (46)] of $G_{ab}$, we have:

$$\bar{G}(t,t') = J^{-1}(t_a,t_b)\left[g(t)/g(t')\right]^{1/2}\Big\{\left[\frac{\dot{g}(t')}{2g(t')} - k(t')\right]\left[J(t_a,t)J(t',t_b)Y(t-t')\right.$$
$$\left. + J(t_a,t')J(t,t_b)Y(t'-t)\right] + J(t_a,t)\,J_{,t'}(t',t_b)Y(t-t')$$
$$+ J_{,t'}(t_a,t')J(t,t_b)Y(t'-t)\Big\} \tag{68}$$

It is readily verified that $\bar{G}$ has a discontinuity of magnitude 1 along the diagonal $t = t'$ : $\left(\lim_{t\to t'} - \lim_{t'\to t}\right)\bar{G}(t,t') = 1$. For this, one only needs (41) and the fact that the Wronskian of $D_1$ and $D_2$ is 1. Q.E.D.

The AA term

$$G_p(t,t') = \int_T dv\, \omega(t,v)\omega(t',v)\left[f(v) - m\frac{k^2(v)}{g(v)}\right]$$
$$-m\int_T \frac{dv}{g(v)}\omega_{,v}(t,v)\omega_{,v}(t',v) \quad - \frac{m}{g(t)}\delta(t-t')$$
$$+m\int_T \frac{dv}{g(v)}k(v)\left[\omega(t',v)\omega(t,v)\right]_{,v} + \frac{2m}{g(t')}\left[\frac{\partial}{\partial t'} - k(t')\right]\omega(t,t')$$
$$= \frac{2m}{g(t')}\left[\frac{\partial}{\partial t'} - k(t')\right]\omega(t,t') - \frac{m}{g(t)}\delta(t-t')$$
$$- \int_T \omega(t,v)\,\underset{\sim}{\mathcal{S}}_{v}\,\omega(t',v)\,dv \qquad (69)$$

Using (63) and (64), we have:

$$\underset{\sim}{\mathcal{S}}_{t}\,\omega(t',t) = \frac{m}{g(t)}\left[k(t') - \frac{\dot{g}(t')}{g(t')} - \frac{\partial}{\partial t'}\right]\delta(t-t'), \qquad (70)$$

which gives

$$G_p(t,t') = \frac{m}{g(t')}\left[\frac{\partial}{\partial t'} - k(t')\right]\omega(t,t') - \frac{m}{g(t)}\delta(t-t')$$
$$= \frac{m}{\sqrt{g(t)}}\left[\frac{1}{2}\frac{\dot{g}(t)}{g(t)} - k(t) + \frac{\partial}{\partial t}\right]\frac{\bar{G}(t,t')}{\sqrt{g(t)}} - \frac{m\,\delta(t-t')}{g(t)}$$
$$= \frac{m}{g(t)}\left[\frac{\partial}{\partial t} - k(t)\right]\bar{G}(t,t') - \frac{m\,\delta(t-t')}{g(t)} \qquad (71)$$
$$= \frac{m^2}{g(t)g(t')}\left[\frac{\partial}{\partial t} - k(t)\right]\left[\frac{\partial}{\partial t'} - k(t')\right]G_{ab}(t,t') - \frac{m\,\delta(t-t')}{g(t)} \qquad (72)$$

Using the specific expression [(64) with (46)] of $G_{ab}$, we find that the $\delta$ function term cancels another similar term (due to the fact that the Wronskian of $D_1$ and $D_2$ is 1) and we are left with the following expression for $G_p$, symmetric and continuous along the diagonal $t = t'$:

$$
\begin{aligned}
G_p(t,t') = \frac{m Y(t-t')}{\sqrt{g(t) g(t')}} J^{-1}(t_a,t_b) \big[ & J(t_a,t) J(t',t_b) \gamma(t) \gamma(t') \\
& + J(t_a,t) J_{,t'}(t',t_b) \gamma(t) + J_{,t}(t_a,t) J(t',t_b) \gamma(t') \\
& + J_{,t}(t_a,t) J_{,t'}(t',t_b) \big] \;. \\
& + t \sim t'
\end{aligned}
\tag{73}
$$

where $\gamma(t) \equiv -k(t) + \dot{g}(t)/2g(t)$ and $F(t,t') + t \sim t' \equiv F(t,t') + F(t',t)$.

<u>The B term</u>

$$
\begin{aligned}
\bar{q}(t) = {} & q_{co}(t) + \int_T dt' \, q_{co}(t') \sigma(t,t') \left[ f(t') - \frac{m k^2(t')}{g(t')} \right] \\
& + m \int_T \frac{dt'}{g(t')} \big[ k(t') q_{co}(t') \sigma_{,t'}(t,t') + \dot{k}(t') \dot{q}_{co}(t') \sigma(t,t') \\
& \qquad - \dot{q}_{co}(t') \sigma_{,t'}(t,t') \big] \\
= {} & q_{co}(t) + m \int_T \frac{dt}{g(t')} \sigma(t,t') \, \underset{\sim}{Q}_{t'} q_{co}(t') = q_{co}(t)
\end{aligned}
\tag{74}
$$

by virtue of (31). Q.E.D.

The A term

$$\bar{p}(t) = \frac{m}{g(t)}\left[\frac{d}{dt} - k(t)\right]q_{co}(t) + \int_T dt'\,\omega(t,t')\,q_{co}(t')\left[f(t') - m\frac{k^2(t')}{g(t')}\right]$$

$$+ m\int_T \frac{dt'}{g(t')}\left[k(t')\omega(t,t')\dot{q}_{co}(t') + k(t')q_{co}(t')\omega_{,t'}(t,t') - \dot{q}_{co}(t')\omega_{,t'}(t,t')\right]$$

$$= \frac{m}{g(t)}\left[\frac{d}{dt} - k(t)\right]q_{co}(t) + m\int_T \frac{dt'}{g(t')}\,\omega(t,t')\,\underset{\sim}{Q}_{t'}q_{co}(t')$$

$$= \frac{m}{g(t)}\left[\frac{d}{dt} - k(t)\right]q_{co}(t)$$

$$= p_{co}(t) \tag{75}$$

by virtue of (31) and the fact that $p_{co} = (\partial L_o/\partial \dot{q})_{q=q_{co}}$ . Q.E.D.

The relations we have derived so far make it simple to verify that $\mathcal{G}(t,t')$ in (19) is indeed a Green function of the small disturbance operator (20) in phase space. The "11" term of the resulting metrix is $\delta(t-t')$ because of (67) and (60). The "12" term is 0 because of (25), (60), and the fact that $(\partial/\partial t + \partial/\partial t')\,\delta(t-t') = 0$ . The "21" term is 0 because of (67). Finally, the "22" term is $\delta(t-t')$ because of (71). The fact that $(\bar{q},\bar{p})$ is the average path will be proved later in the paper.

Example 1: The Free Particle

For a free particle, $k(t) = f(t) = 0$ , $g(t) = 1$ , $H_o = p^2/2m$ , $\underset{\sim}{S} = -m\,d^2/dt^2$ , $D_1(t) = 1$ , $D_2(t) = t_b - t$ , $J(t,t') = t' - t$ , and $M = m/T$ . The covariance of the corresponding measure in phase space is (19), where:

$$G_{ab}(t,t') = \frac{(t'-t_a)(t_b-t)Y(t-t') + (t-t_a)(t_b-t')Y(t'-t)}{mT} \tag{76}$$

$$\bar{G}(t,t') = \frac{(t_b-t)Y(t-t') - (t-t_a)Y(t'-t)}{T} \tag{77}$$

$$G_p(t,t') = -\frac{m}{T} \qquad (T \equiv t_b - t_a) \tag{78}$$

We have $\underset{\sim}{\mathcal{S}}\, G_{ab}(t,t') = \delta(t-t')$ . The average position and momentum are the classical ones:

$$q_{co}(t) = \frac{q_b(t-t_a) + q_a(t_b-t)}{T} \tag{79}$$

$$p_{co}(t) = \frac{m(q_b - q_a)}{T} \tag{80}$$

The Wiener measure for a free particle in Brownian motion, defined on the configuration space of paths $\mathcal{C}_- \equiv \{q(t) \text{ on } T = [t_a, t_b] \mid q(t_a) = 0,\ q(t_b) \text{ unrestricted}\}$ can be readily extended to the phase space $\mathcal{P}_-$ defined by

$$\mathcal{P}_- \equiv \{[p(t), q(t)] \text{ on } T = [t_a, t_b] \mid q(t_a) = 0,\ q(t_b) \text{ and } p(t) \text{ unrestricted}\} \tag{81}$$

by letting $t_b \to \infty$ . The covariance $\mathcal{G}_-(t,t')$ is then

$$\mathcal{G}_-(t,t') = \begin{pmatrix} \frac{1}{m}[(t'-t_a)Y(t-t') + (t-t_a)Y(t'-t) & Y(t-t') \\ Y(t'-t) & 0 \end{pmatrix} \tag{82}$$

The "11" term is a (symmetric) Green function of the small disturbance operator

$-m\,d^2/dt^2$ such that $G(t_a,t)=0$. It is discussed in Refs. 4-6.

Example 2: The Harmonic Oscillator

For a harmonic oscillator, $k(t)=0$, $g(t)=1$, $f(t)=m\omega^2$, $H_0=p^2/2m+m\omega^2q^2/2$, $\underset{\sim}{\mathcal{D}}=-m(d^2/dt^2+\omega^2)$, $D_1(t)=\cos\omega(t_b-t)$, $D_2(t)=\omega^{-1}\sin\omega(t_b-t)$, $J(t,t')=\omega^{-1}\sin\omega(t'-t)$, and $M=m\omega/\sin\omega T$. The covariance of the corresponding measure in phase space is (19), with:

$$G_{ab}(t,t')=\frac{\sin\omega(t_b-t')\sin\omega(t-t_a)Y(t'-t)+\sin\omega(t_b-t)\sin\omega(t'-t_a)Y(t-t')}{m\omega\sin\omega T} \tag{83}$$

$$\bar{G}(t,t')=\frac{\sin\omega(t_b-t)\cos\omega(t'-t_a)\,Y(t-t')-\cos\omega(t_b-t')\sin\omega(t-t_a)\,Y(t'-t)}{\sin\omega T} \tag{84}$$

$$G_p(t,t')=\frac{-m\omega}{\sin\omega T}\left[\cos\omega(t_b-t)\cos\omega(t'-t_a)Y(t-t')+\cos\omega(t_b-t')\cos\omega(t-t_a)Y(t'-t)\right] \tag{85}$$

We have $\underset{\sim}{\mathcal{D}}\,G_{ab}(t,t')=\delta(t-t')$. The average position and momentum are the classical ones:

$$q_{co}(t)=(\sin\omega T)^{-1}\left[q_a\sin\omega(t_b-t)+q_b\sin\omega(t-t_a)\right] \tag{86}$$

$$p_{co}(t)=m\omega(\sin\omega T)^{-1}\left[q_b\cos\omega(t-t_a)-q_a\cos\omega(t_b-t)\right] \tag{87}$$

## III. PATH INTEGRATION IN PHASE SPACE

We now show how to carry out the path integral of a cylindrical functional with respect to an arbitrary measure in phase space given by its Fourier transform. A cylindrical functional is one which depends on only a finite number of terms of the form $\langle \mu, q \rangle$ or $\langle \nu, p \rangle$, i.e. $\int_T q(t)\, d\mu(t)$ or $\int_T p(t)\, d\nu(t)$.

THEOREM 2. Let $w$ be a measure in phase space $\mathcal{P}$ defined by its Fourier transform $\mathcal{F}w(\mu,\nu)$. A cylindrical functional $F$ on $\mathcal{P}$ can be integrated over $\mathcal{P}$ with respect to the measure $w$ by reducing it to an ordinary integral as follows:

$$
\begin{aligned}
I &\equiv \int_{\mathcal{P}} F(\langle \mu_1, q\rangle, \dots, \langle \mu_n, q\rangle, \langle \nu_1, p\rangle, \dots, \langle \nu_m, p\rangle)\, dw(p,q) \\
&= \int_{\mathbb{R}^{n+m}} F(u_1, \dots, u_n, v_1, \dots, v_m)\, du_1 \dots du_n\, dv_1 \dots dv_m\, (2\pi)^{-n-m} \\
&\quad \times \int_{\mathbb{R}^{n+m}} \mathcal{F}w(\xi^1 \mu_1 + \dots + \xi^n \mu_n, \eta^1 \nu_1 + \dots + \eta^m \nu_m) \, \cdot \\
&\quad \times \exp i(\xi^1 u_1 + \dots + \xi^n u_n + \eta^1 v_1 + \dots + \eta^m v_m)\, d\xi^1 \dots d\xi^n\, d\eta^1 \dots d\eta^m
\end{aligned}
\tag{88}
$$

Proof

This proof is similar to the ones used for similar formulas in configuration space path integrals without limiting procedure[5,7]. Consider the linear continuous mapping $P_{n,m}$ :

$$P_{n,m} : \mathcal{P} \to \mathbb{R}^{n+m} \text{ by } (q,p) \mapsto (u,v) \text{ where } \begin{cases} u_i = \langle \mu_i , q \rangle \text{ for } i = 1 \text{ to } n \\ v_j = \langle \nu_j , p \rangle \text{ for } j = 1 \text{ to } m \end{cases} \tag{89}$$

Under this mapping, we have

$$I = \int_{\mathbb{R}^{n+m}} F(u_1, \ldots, u_n, v_1, \ldots, v_m)\, dw_{P_{n,m}}(u,v) \tag{90}$$

where $w_{P_{n,m}}$ is the image of $w$ under $P_{n,m}$. This image is a measure in $\mathbb{R}^{n+m}$. By theorem[3], $\mathcal{F}w_{P_{n,m}}(\xi,\eta) = \mathcal{F}w[\tilde{P}_{n,m}(\xi,\eta)]$, where $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$, and $\tilde{P}_{n,m}$ is the transpose mapping from $\mathbb{R}^{n+m}$ to $\mathcal{M}$, the space of bounded measures on the time interval $T = [t_a, t_b]$. We have:

$$\begin{aligned} \langle \tilde{P}_{n,m}(\xi,\eta) ; (q,p) \rangle &\equiv \langle (\xi,\eta) ; P_{n,m}(q,p) \rangle = \langle (\xi,\eta) ; (u,v) \rangle \\ &= \xi \cdot u + \eta \cdot v = \xi^i \langle \mu_i , q \rangle + \eta^j \langle \nu_j , p \rangle \\ &= \langle (\xi^i \mu_i , \eta^j \nu_j) ; (q,p) \rangle \end{aligned} \tag{91}$$

and hence $\tilde{P}_{n,m}(\xi,\eta) = (\xi^i \mu_i , \eta^j \nu_j)$. Therefore:

$$\begin{aligned} dw_{P_{n,m}}(u,v) &= \mathcal{F}^{-1}_{\xi,\eta}\left[ \mathcal{F}w(\xi^i \mu_i , \eta^j \nu_j) \right] \\ &= (2\pi)^{-n-m}\, du_1 \ldots du_n\, dv_1 \ldots dv_m \\ &\quad \times \int_{\mathbb{R}^{n+m}} \exp(i\xi^i u_i + i\eta^j v_j)\, \mathcal{F}w(\xi^i \mu_i , \eta^j \nu_j)\, d\xi^1 \ldots d\xi^n d\eta^1 \ldots d\eta^m . \end{aligned} \tag{92}$$

Q.E.D.

Corollary 1

If $F$ depends only on $p$ (resp. $q$ ), the path integral reduces to an integral over momentum (resp. configuration) space. In compressed notation:

$$\int_{\mathcal{P}} F(\langle \nu, p \rangle)\, dw(p,q) = \int_{\mathbb{R}^m} \frac{dv}{(2\pi)^m} F(v) \int_{\mathbb{R}^m} \mathcal{F}w(0, \eta . \nu)\, e^{i\eta . v}\, d\eta \qquad (93)$$

$$\int_{\mathcal{P}} F(\langle \mu, q \rangle)\, dw(p,q) = \int_{\mathbb{R}^n} \frac{du}{(2\pi)^n} F(u) \int_{\mathbb{R}^n} \mathcal{F}w(\xi . \mu, 0)\, e^{i\xi . u}\, d\xi \qquad (94)$$

Thus, in the second case, the measure $w(p,q)$ in phase space has the same effect as the measure *$w_{ab}(q)$ in the configuration space $\mathcal{C}_{ab}$ of paths* such that $q(t_a) = q_a$ and $q(t_b) = q_b$ , i.e.

$$\int_{\mathcal{P}} F(\langle \mu, q \rangle)\, dw(p,q) = \int_{\mathcal{C}_{ab}} F(\langle \mu, q \rangle)\, dw_{ab}(q) \qquad (95)$$

$\mathcal{C}_{ab}$ and $w_{ab}$ were introduced and studied in Ref. 7.

Moments formula

$$\int_{\mathcal{P}} \langle \mu_1, q \rangle \cdots \langle \mu_n, q \rangle \langle \nu_1, p \rangle \cdots \langle \nu_m, p \rangle\, dw(p,q)$$

$$= i^{m+n} \frac{\partial^{m+n}}{\partial \xi^1 \ldots \partial \xi^n \partial \eta^1 \ldots \partial \eta^m} \mathcal{F}w(\xi^1 \mu_1 + \ldots + \xi^n \mu_n, \eta^1 \nu_1 + \ldots + \eta^m \nu_m) \Big|_{\xi = \eta = 0} \qquad (96)$$

Proof

Theorem 2 and the fact that $\int_{\mathbb{R}} x e^{ikx}\, dx = -2\pi i\, \delta'(x)$ are needed.

Application to the Gaussian Measure

If we apply Theorem 2 to the Gaussian measure defined in (12) in Theorem 1, we obtain:

$$\int_{\mathcal{P}} F(\langle \mu_1, q\rangle, \ldots, \langle \mu_n, q\rangle, \langle \nu_1, p\rangle, \ldots, \langle \nu_m, p\rangle\, dw(p,q)$$

$$= \int_{\mathbb{R}^{n+m}} \frac{F(u_1, \ldots, u_n, v_1, \ldots, v_m)\, du_1 \ldots du_n\, dv_1 \ldots dv_m}{\left[(2\pi i \hbar)^{m+n} \det W \cdot \det S\right]^{1/2}}$$

$$\times \exp \frac{i}{2\hbar} \Big\{ (S^{-1})^{ij} (v_i - b_i)(v_j - b_j) - 2\,(W^{-1} C S^{-1})^{ij} (u_i - a_i)(v_j - b_j) + (W^{-1} + W^{-1} C S^{-1} \tilde{C} W^{-1})^{ij} (u_i - a_i)(u_j - a_j) \Big\} \qquad (97)$$

where

$$a_i \equiv \langle \mu_i, \bar{q} \rangle \qquad (98)$$

$$b_i \equiv \langle \nu_i, \bar{p} \rangle \qquad (99)$$

$$W_{ij} \equiv \int_T \int_T G_{ab}(t,t')\, d\mu_i(t)\, d\mu_j(t') \qquad (n \times n) \qquad (100)$$

$$C_{ij} \equiv \int_T \int_T \bar{G}(t,t')\, d\mu_i(t)\, d\nu_j(t') \qquad (n \times m) \qquad (101)$$

$$V_{ij} \equiv \int_T \int_T G_p(t,t')\, d\nu_i(t)\, d\nu_j(t') \qquad (m \times m) \qquad (102)$$

$$S \equiv V - \tilde{C} W^{-1} C \qquad (m \times m) \qquad (103)$$

$\tilde{C}$ being the transpose of $C$.

Proof

The proof is straightforward with repeated use of the formula[14]

$$\int_{\mathbb{R}^n} \varphi(b^i u_i) \exp\left(-\tfrac{1}{2} A^{ij} u_i u_j\right) du_1 \cdots du_n \qquad (104)$$

$$= \frac{(\sqrt{2\pi})^{n-1}}{|c|\sqrt{\det A}} \int_{\mathbb{R}} \varphi(u) \exp\left(\frac{-u^2}{2c^2}\right) du \quad , \quad \mathrm{Re}(A^{ij}u_i u_j) \geq 0 \ \forall u \in \mathbb{R}^n$$

where $c^2 \equiv b^i b^j (A^{-1})_{ij}$ . Here $\varphi(u) = e^{iu}$ and one needs

$$\int_{\mathbb{R}} \exp(ax^2 + bx)\, dx = (-\pi/a)^{1/2} \exp(-b^2/4a) \ , \ \mathrm{Re}(a) \leq 0 \qquad (105)$$

If the functional to be integrated does not have a $p$ dependence, then Corollary 1 gives:

$$\int_{\mathcal{P}} F(\langle \mu_1, q\rangle, \cdots, \langle \mu_n, q\rangle)\, dw(p,q)$$

$$= \int_{\mathcal{C}_{ab}} F(\langle \mu_1, q\rangle, \cdots, \langle \mu_n, q\rangle)\, dw_{ab}(q) \qquad (106)$$

$$= \int_{\mathbb{R}^n} \frac{F(u_1, \ldots, u_n)\, du_1 \cdots du_n}{(2\pi i \hbar)^{n/2} (\det W)^{1/2}} \exp\left\{\frac{i}{2\hbar} (W^{-1})^{ij} (u_i - a_i)(u_j - a_j)\right\}$$

which is formula (59) in Ref. 7. If $F$ has no $q$ dependence, then:

$$\int_{\mathcal{P}} F(\langle \nu_1, p\rangle, \cdots, \langle \nu_m, p\rangle)\, dw(p,q)$$

$$= \int_{\mathbb{R}^m} \frac{F(v_1, \ldots, v_m)\, dv_1 \cdots dv_m}{(2\pi i \hbar)^{m/2} (\det V)^{1/2}} \exp\left\{\frac{i}{2\hbar} (V^{-1})^{ij} (v_i - b_i)(v_j - b_j)\right\} \qquad (107)$$

Averages and covariances

The moments formula (96) applied to the Gaussian measure $w$ readily gives the average position and momentum for $w$ :

$$\int_{\mathcal{P}} q(t)\, dw(p,q) = \bar{q}(t) \tag{108}$$

$$\int_{\mathcal{P}} p(t)\, dw(p,q) = \bar{p}(t) \tag{109}$$

indicating that $\bar{p}$ and $\bar{q}$ were correctly identified in the statement of Theorem 1. The covariances are:

$$\int_{\mathcal{P}} [q(t) - \bar{q}(t)][q(t') - \bar{q}(t')]\, dw(p,q) = i\hbar\, G_{ab}(t,t') \tag{110}$$

$$\int_{\mathcal{P}} [p(t) - \bar{p}(t)][p(t') - \bar{p}(t')]\, dw(p,q) = i\hbar\, G_{p}(t,t') \tag{111}$$

$$\int_{\mathcal{P}} [q(t) - \bar{q}(t)][p(t') - \bar{p}(t')]\, dw(p,q) = i\hbar\, \bar{G}(t,t') \tag{112}$$

$\bar{G}(t,t')$ is the only covariance not to be continuous across the diagonal $t = t'$. It has a jump of magnitude 1 there, as established earlier. Thus, the correlation between $p$ and $q$ at a given time $t$ with respect to the measure $w$ can only be established to within $i\hbar$.

The Set of "Important Paths"

The variances $i\hbar\, G_{ab}(t,t)$ and $i\hbar\, G_p(t,t)$, squares of the "standard deviations" $\Delta q(t)$ and $\Delta p(t)$, provide a measure of the degree of dispersion of the Feynman paths about the average position and momentum. We now calculate $\Delta q$ and $\Delta p$ for the free particle and the harmonic oscillator, using the results established earlier for these two systems.

Free particle

$$\Delta q(t) = \left[ i\hbar \frac{(t-t_a)(t_b-t)}{mT} \right]^{1/2} \tag{113}$$

$$\Delta p(t) = \left[ -\frac{i\hbar m}{T} \right]^{1/2} \tag{114}$$

$$(\Delta p \cdot \Delta q)(t) = \frac{\hbar}{T} \left[ (t-t_a)(t_b-t) \right]^{1/2} \tag{115}$$

Harmonic oscillator

$$\Delta q(t) = \left[ \frac{i\hbar \sin\omega(t_b-t) \sin\omega(t-t_a)}{m\omega \sin\omega T} \right]^{1/2} \tag{116}$$

$$\Delta p(t) = \left[ \frac{-i\hbar m\omega \cos\omega(t_b-t) \cos\omega(t-t_a)}{\sin\omega T} \right]^{1/2} \tag{117}$$

$$(\Delta p . \Delta q)(t) = \frac{\hbar}{2|\sin\omega T|} \left[ \sin 2\omega(t-t_a) \sin 2\omega(t_b-t) \right]^{1/2} \tag{118}$$

In both instances, we have:

$$(\Delta p . \Delta q)(t) \leqslant \frac{\hbar}{2} \tag{119}$$

A first glance at this relation might give the impression that we have obtained the uncertainty principle backwards. In fact, this relation has nothing to do with the uncertainty principle. If $\Delta q$ and $\Delta p$ are calculated with respect to $\Psi(q,t)$ and $\Phi(p,t)$, the wave functions of the particles in configuration and momentum spaces at time $t$, then they reflect the effect of measurement, and $(\Delta p . \Delta q)(t) \geqslant \hbar/2$. But if $\Delta q$ and $\Delta p$ are calculated with respect to the phase space measure $W(p,q)$, then they simply reflect which paths are weighed more heavily (i.e. contribute the most) in the sum over paths. To be more precise, they determine how far one must deviate from the average (here, classical) path to still find paths which contribute appreciably to the sum over paths. In these two cases (as in most cases), these "important" paths are so close to the average path in phase space that $\Delta p \cdot \Delta q$ is always extremely small -- in fact, never larger than $\hbar/2$ .

Note that the average square velocity in configuration space is infinite, indicating that the "important" paths in configuration space are the nondifferentiable ones, a well-known result. For example, in the case of the free particle,

$$\left[(\Delta\dot{q})(t)\right]^2 \equiv \int_{\mathcal{P}} \left[\dot{q}(t) - \dot{\bar{q}}(t)\right]^2 dw(p,q) = \lim_{t\to t'} \frac{\partial^2}{\partial t \partial t'} \int_{\mathcal{P}} [q(t) - \bar{q}(t)][q(t') - \bar{q}(t')]\, dw(p,q)$$

$$= \lim_{t\to t'} \frac{\partial^2}{\partial t \partial t'}\, i\hbar G_{ab}(t,t') = \lim_{t\to t'} \left[-\frac{i\hbar m}{T} + i\hbar m\,\delta(t-t')\right]$$

$$\to \infty \qquad (120)$$

Comparison with (114) shows that we do not have $\Delta p(t) = \Delta m\dot{q}(t)$; nor should we expect it, since no relationship is assumed between $p$ and $q$ in the unrestricted sum over paths in phase space.

## IV. APPLICATIONS

### 1. Perturbation Expansion

The propagator corresponding to $\underset{\sim}{H} = \underset{\sim}{H}_0 + \alpha \underset{\sim}{H}_1$ , where $\underset{\sim}{H}_0$ is given by (15), is:

$$\langle q_b, t_b | q_a, t_a \rangle = K_0 \int_{\mathcal{P}} \exp\left[-\frac{i\alpha}{\hbar}\int_{t_a}^{t_b} H_1[p(t), q(t), t]\, dt\right] dw(p,q) \qquad (121)$$

where $H_1$ is the classical equivalent[1] of $\underset{\sim}{H}_1$ , and $w$ is defined in (12).

This is a direct application of Theorem 1. By expanding the exponential and carrying out the resulting path integrals by use of the moments formula (96), one obtains the propagator as a power series in $\alpha$ .

2. Semiclassical Expansion

A more useful application of Theorem 1 is to use it to expand the ratio of the propagator to its WKB approximation in a power series in $\hbar$ :

$$K = K_{WKB} \left( 1 + \hbar K_1 + \hbar^2 K_2 + \dots \right) \tag{122}$$

This is the semiclassical expansion, treated in configuration space in Refs. 6, 8, and 9. The terms $K_i$ are "doable" path integrals of cylindrical functionals, which can be evaluated using (96). Such an expansion has been used to shed some light on the anharmonic oscillator[6,9]. Sometimes, due to the peculiarities of the Hamiltonian, a configuration space path integral scheme is not possible. A phase space path integral scheme is always possible. We now show how to generalize the path integral treatment of the semiclassical expansion to arbitrary Hamiltonians.

THEOREM 3. The propagator corresponding to an arbitrary $H$ [ see Eq. (1)] can be expressed as the following path integral:

$$K = K_{WKB} \int_{\mathcal{P}_0} \exp\left\{ \frac{i}{\hbar} \Omega(q_c, p_c)(x,y) \right\} dw(y,x) \tag{123}$$

where

$$(1)\quad K_{WKB} = \left( \frac{-1}{2\pi i \hbar} \frac{\partial^2 S(q_c, p_c)}{\partial q_a \, \partial q_b} \right) \exp\left[ \frac{i}{\hbar} S(q_c, p_c) \right] \tag{124}$$

(2) $(x,y) \in \mathcal{P}_0 \equiv \{ [x(t), y(t)] \text{ on } T \equiv [t_a, t_b] \mid x(t_a) = x(t_b) = 0,$

$$y(t) \text{ unrestricted} \} \tag{125}$$

(3) $\Omega(q_c, p_c)$ is an operator resulting from the expansion of the action functional about the classical path $(q_c, p_c)$ :

$$\begin{aligned} S(q,p) &\equiv S(x+q_c, y+p_c) \\ &= S(q_c,p_c) + S'(q_c,p_c)(x,y) + \frac{1}{2!} S''(q_c,p_c)(x,y) \\ &\quad + \Omega(q_c,p_c)(x,y) \end{aligned} \tag{126}$$

(4) The Gaussian measure $w$ is as in Theorem 1, with

$$\frac{g(t)}{m} = \left. \frac{\partial^2 H}{\partial p^2} \right|_{\substack{q=q_c \\ p=p_c}} \tag{127}$$

$$f(t) = \left. \frac{\partial^2 H}{\partial q^2} \right|_{\substack{q=q_c \\ p=p_c}} \tag{128}$$

$$k(t) = \left. \frac{\partial^2 H}{\partial q \partial p} \right|_{\substack{q=q_c \\ p=p_c}} \tag{129}$$

The path integral can be evaluated by expanding the exponential in a power series, which can then be rearranged to yield a power series in $\hbar$ where the terms depend only on the classical path $(q_c, p_c)$.

Proof

In the expansion (126) of the action, the term $S'(q_c,p_c)(x,y)$ is 0 by definition of the classical path $(q_c, p_c)$ (it yields Hamilton's equations). The term $S''(q_c,p_c)(x,y)/2$ is

$$\frac{1}{2}(x,y)\,\underset{\sim}{\mathcal{O}}\begin{pmatrix}x\\y\end{pmatrix} \tag{130}$$

where $\underset{\sim}{\mathcal{O}}$ is the small disturbance operator (59). Expanding this term, integrating the $-\frac{1}{2}\int_T x(t)\,\dot{y}(t)\,dt$ term by parts to get $\frac{1}{2}\int_T y(t)\,\dot{x}(t)\,dt$, and lumping the resulting expression (quadratic in $x$ and $y$) into the measure by using Theorem 1, yields the desired result. The $\hbar$ in the denominator will always be cancelled by higher powers of $\hbar$ in the numerator, due to the fact that the moments formula needed to evaluate the various path integrals arising in the expansion of the exponential in (123) yields products of covariances, each of which is multiplied by $\hbar$ [6,8,9].

## V. CONCLUSION

The generalization of the path integral scheme to arbitrary Hamiltonians, which can only be done in phase space, is best carried out without the limiting process which makes the integrals difficult to compute. This paper has built Gaussian phase space measures which do not require any reference to such a limiting process, shown how to integrate with respect to them, and given examples of how these measures can be of use in solving problems. It would be useful to find non-Gaussian measures, which would absorb larger parts of the functionals to be integrated.

## VI. ACKNOWLEDGMENT

I thank Dr. W. Hurley for a discussion.

FOOTNOTES

1. In most cases of physical interest, e.g. when the quantum Hamiltonian operator is $\underset{\sim}{H} = [\underset{\sim}{P} - (e/c)A(\underset{\sim}{Q})]^2/2m + e\,\Phi(\underset{\sim}{Q})$ , $H$ in (1) is the classical Hamiltonian $H_c$ . For stronger couplings of $\underset{\sim}{Q}$ and $\underset{\sim}{P}$ , e.g. when there is a metric $g(\underset{\sim}{Q})$ to be considered, $H$ in (1) is of the form $H_c + O(\hbar^2)$ . In M. M. Mizrahi, J. Math. Phys. 16 (1975), 2201-2206, it was shown that $H$ could be the Weyl transform of $\underset{\sim}{H}$ . Other possibilities, all yielding the same propagator, are being investigated. At any rate, we always have $H = H_c + O(\hbar^2)$ for Hermitian $\underset{\sim}{H}$s . In this paper, we assume that $H = H_c$ .

2. See, e.g., R. P. Feynman, Phys. Rev. 84 (1951), 108-128, Appendix B; H. Davies, Proc. Camb. Phil. Soc. 59 (1963), 147-155; C. Garrod, Rev. Mod. Phys. 38 (1966), 483-494; and M. M. Mizrahi, op. cit.

3. See, e.g., N. Bourbaki, Eléments de Mathématiques (Hermann, Paris, 1969), Vol. XXXV, Book VI, Chap. IX.

4. C. DeWitt-Morette, Comm. Math. Phys. 28 (1972), 47-67.

5. C. DeWitt-Morette, Comm. Math. Phys. 37 (1974), 63-81.

6. M. M. Mizrahi, "An Investigation of the Feynman Path Integral Formulation of Quantum Mechanics", Ph.D. dissertation, the University of Texas at Austin, August 1975, unpublished.

7. M. M. Mizrahi, J. Math. Phys. 17 (1976), 566-575.

8. C. DeWitt-Morette, Ann. of Phys. 97 (1976), 367-399.

9. M. M. Mizrahi, "WKB Expansions by Path Integrals, With Applications to the Anharmonic Oscillator", preprint, University of Texas at Austin and Center for Naval Analyses of the University of Rochester.

10. Only in the case of the Wiener integral (no "i"s in the exponent) is a bona fide measure obtained. In the case of the Feynman path integral, the imaginary Gaussian measures on $\mathbb{R}^n$ , building blocks of the promeasure one hopes to turn into a measure, are not bounded. This fact makes this attempt at mathematical legalization fall through. However, when one works with the Fourier transforms of the promeasure, the boundedness requirement is no longer needed, and progress can be made for computational purposes. C. DeWitt-Morette calls the resulting objects "pseudomeasures", P. Krée [Bull. Soc. Math. France 46 (1976), 143-162] calls them "prodistributions". For simplicity we call them "measures", as they are formally used as such.

11. M. M. Mizrahi, J. Math. Phys. 16 (1975), 2201-2206.

12. This is a very simple application of more general restrictions on the use of the given WKB approximation formula to a certain class of correspondence rules between the classical and quantum Hamiltonians, found in M. M. Mizrahi, J. Math. Phys. 18 (1977), 786-790.

13. In "Path Integration in Phase Space", by C. DeWitt-Morette, A. Maheshwari, and B. Nelson, preprint (to appear in Gen. Rel. and Grav.), a similar measure is presented using a different approach. This paper and the present one, written independently, complement each other and should be read concurrently.

14. This formula can be proved by path integrals -- see Ref. 7.

Addenda to M.M. Mizrahi's "Phase Space Path Integrals, Without Limiting Procedure"

## APPENDIX A

Intuitive Justification of Theorem 1 and of Path Integration Without Limiting Procedure

A Gaussian measure on $\mathbb{R}^n$ which can be written as

$$d\gamma(y) = (2\pi)^{-n/2}(\det C)^{1/2}\exp\left(-\frac{1}{2}C^{ij}y_i y_j\right)dy_1 \cdots dy_n \qquad \text{(A1)}$$

has as its Fourier transform the exponential of a quadratic form involving the inverse of the matrix $C$:

$$(\mathcal{F}\gamma)(x) \equiv \int_{\mathbb{R}^n} e^{-ix\cdot y}\,d\gamma(y) = \exp\left[-\frac{1}{2}(C^{-1})_{ij}x^i x^j\right] \qquad \text{(A2)}$$

How does this carry over to infinite-dimensional spaces? This is the question answered in Theorem 1. The phase space measure $w(p,q)$ in (11), after integrations by parts, can be written as:[15]

$$dw(p,q) \sim K_0^{-1}\left[\frac{dp\,dq}{2\pi\hbar}\right]\exp\frac{i}{2\hbar}\left\{\int_T (q(t)\;\; p(t))\,\underset{\sim}{\mathcal{O}}\begin{pmatrix} q(t) \\ p(t)\end{pmatrix}dt + [q_b\,p(t_b) - q_a\,p(t_a)]\right\}, \qquad \text{(A3)}$$

where $\underset{\sim}{\mathcal{O}}$ is the operator defined in (20). $\underset{\sim}{\mathcal{O}}$ is seen to play the role of the matrix $C$ above. Therefore, by analogy, one expects the Fourier transform of $w$ to be the exponential of a quadratic form involving the inverse of $\underset{\sim}{\mathcal{O}}$, i.e., one of the Green functions of $\underset{\sim}{\mathcal{O}}$, so that

$$\int_T \underset{\sim}{\mathcal{O}}_t\,\mathcal{G}(t,t')\,dt' = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}. \qquad \text{(A4)}$$

This is precisely what was proved. Which Green function is used depends on the path space considered (for example, in the free particle case we saw that consideration of the space $\mathcal{P}_-$ instead of $\mathcal{P}$ led to a Green function $\mathcal{G}_-$ different from $\mathcal{G}$ ). The terms involving $q_a$ and $q_b$ enter when the average path is non-zero. The reason why $\underset{\sim}{\mathcal{O}}$ was called the small disturbance operator in Theorem 1 is that this is what it is when the action is expanded about the classical path, as we have seen.

To illustrate this intuitive justification further, we consider the free-particle measure $w_{ab}$ in configuration space. It is ( $N$ is the "normalization" necessary in the time-slicing approach):

$$dw_{ab}(q) \sim \left[\frac{dq}{N}\right] \exp\left\{\frac{im}{2\hbar}\int_T \dot{q}^2(t)dt\right\}, \tag{A5}$$

which can be rewritten as:

$$dw_{ab}(q) \sim \left[\frac{dq}{N}\right] \exp\left\{\frac{i}{2\hbar}\int_T q(t)\left(-m\frac{d^2}{dt^2}\right)q(t)\,dt + \frac{im}{2\hbar}\left[q_b\,\dot{q}(t_b) - q_a\,\dot{q}(t_a)\right]\right\}. \tag{A6}$$

The Fourier transform of $w_{ab}$ indeed has as its covariance an inverse of $-md^2/dt^2$, namely the Green function $G_{ab}$ in (76)[16]. The form (A6) can be easily generalized to the more general configuration space measure $w_{ab}(q)$ induced by $w(p,q)$ in (11) : $-m\,d^2/dt^2$ is replaced by $\underset{\sim}{\mathcal{S}}$ in (22)-- as proved in (121a)-- and the covariance is $G_{ab}$, the Green function of $\underset{\sim}{\mathcal{S}}$ introduced in theorem 1.

[15]A double integral, corresponding to the double summation in (A1), can be easily obtained by replacing $(q(t)\ \ p(t))$ in (A3) by $\int_T \delta(t-t')(q(t')\ \ p(t'))\,dt'$.

[16]In the case of the free particle in momentum space, a rare case where a measure in momentum space alone can be used, we have

$$dw(p) \sim \left[\frac{dp}{N'}\right] \exp\left[\frac{i}{2m\hbar}\int_T p^2(t)\,dt\right].$$

The operator corresponding to $C$ is then simply the constant $m^{-1}$. Its inverse in the sense of (A4) is the constant $m/T$. It is the negative of $G_p(t,t')$ for the free particle [Eq. (78)] because $p^2/2m$ appears with a different sign in (11).

Insert p. 27 after line 3.

Calculation of $K_0$

$K_0$, the propagator corresponding to $\underset{\sim}{H}_0$ in (15), is given exactly by its WKB approximation. Thus, we only need to calculate the classical action. The action functional is:

$$S_0[q] = \int_T L_0(q,\dot q,t)\,dt = \int_T \left\{ \frac{m}{2g(t)}\left[\dot q(t) - k(t)q(t)\right]^2 - \frac{1}{2}f(t)q^2(t)\right\} dt$$

$$= \frac{1}{2}\int_T q(t)\,\underset{\sim}{\delta}\,q(t)\,dt + \frac{m}{2}\left\{ \frac{q_b}{g(t_b)}\left[\dot q(t_b) - k(t_b)q_b\right] \right. \qquad (121a)$$

$$\left. - \frac{q_a}{g(t_a)}\left[\dot q(t_a) - k(t_a)q_a\right]\right\}$$

where $\underset{\sim}{\delta}$ is the operator (22). This can be easily established by integrations by parts of the $\dot q^2$ and $(q^2)^{\cdot}$ terms. At the classical path $q_{co}$, the first term vanishes since $\underset{\sim}{\delta} q_{co} = \underset{\sim}{Q}\, q_{co} = 0$, and only the integrated term remains. (42) gives $q_{co}$ in terms of the kernel $J$, and we get:

$$S_0[q_{co}] = \frac{m}{2}\left\{ \frac{q_b^2}{g(t_b)}\left[\frac{J_{,2}(t_a,t_b)}{J(t_a,t_b)} + \frac{\dot g(t_b)}{2g(t_b)} - k(t_b)\right]\right. \qquad (121b)$$

$$\left. - \frac{q_a^2}{g(t_a)}\left[\frac{J_{,1}(t_a,t_b)}{J(t_a,t_b)} + \frac{\dot g(t_a)}{2g(t_a)} - k(t_a)\right] - \frac{2q_aq_b}{J(t_a,t_b)\sqrt{g(t_a)g(t_b)}}\right\},$$

where we have used the fact that $J_{,2}(t_a,t_a) = -J_{,1}(t_b,t_b) = 1$. Note that $J_{,2}(t_a,t_b) = D_1(t_a)$ and that $J_{,1}(t_a,t_b) = \dot D_2(t_a)$ ($J_{,i}$ denotes derivative with respect to ith argument). Finally:

$$K_0 = \left[\frac{m}{2\pi i\hbar J(t_a,t_b)}\right]^{1/2} g^{-1/4}(t_a)\,g^{-1/4}(t_b)\exp\left\{\frac{i}{\hbar}S_0[q_{co}]\right\}. \qquad (121c)$$

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